SOME RESULTS ON LYUBEZNIK'S F-MODULES

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ABSTRACT. Let $(R, \mathfrak{m}, K)$ be a regular local ring of equal characteristic $p > 0$. In [Hoc07], Hochster showed that the category of F_R-modules has enough injectives, so that every F_R-module has an injective resolution in this category. We show in this paper that when R is F-finite, every F_R-module has an F_R-injective resolution of length $\leq d+1$ where $d = \dim R$ (or equivalently, $\mathrm{Ext}^i_{F_R}(M, N) = 0$ for all F_R-modules M and N when $i > d+1$). In some sense, this is saying that the category of F_R-modules has "global dimension" $\leq d+1$. In [Hoc07], Hochster also showed that when M and N are F_R-finite F_R-modules, $\mathrm{Hom}_{F_R}(M, N)$ is finite. We show that in general $\mathrm{Ext}^1_{F_R}(M, N)$ is not necessarily finite, but it is finite if M is supported only at $\mathfrak{m}$ and K is separably closed.

1. INTRODUCTION

In [Hoc07], Hochster showed some properties of Lyubeznik's F-modules:

Theorem 1.1 (*cf.* Theorem 3.1 in [Hoc07])**.** *The category of F_R-modules over a Noetherian regular ring R of prime characteristic $p > 0$ has enough injectives, i.e., every F_R-module can be embedded in an injective F_R-module.*

Theorem 1.2 (*cf.* Theorem 5.1 and Corollary 5.2(b) in [Hoc07])**.** *Let R be a Noetherian regular ring of prime characteristic $p > 0$. Let M and N be F_R-finite F_R-modules. Then $\mathrm{Hom}_{F_R}(M, N)$ is a finite-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$ and, hence, is a finite set. Moreover, when R is local, every F_R-finite F_R-module has only finitely many F_R-submodules.*

The main purpose of this paper is to get some further results based on Hochster's results. In connection with Theorem 1.1, we prove that every F_R-module has an F_R-injective resolution of length $\leq d+1$ where $d = \dim R$. In some sense, this is saying that the global dimension of Lyubeznik's F_R-module is finite (and this can be viewed as an analogue of the corresponding statement for $\mathscr{D}$-modules in equal characteristic 0). Theorem 1.2 makes it quite natural to ask whether the higher Ext groups are also finite in this category (when M and N are F_R-finite F_R-modules). We show that in general this fails even for Ext^1 but we prove a special case for finiteness of Ext^1.

This paper is organized as follows. In Section 2 we review some definitions and basic properties of right $R\{F\}$-modules and Lyubeznik's F_R-modules. In Section 3 we relate Lyubeznik's F_R-modules with right $R\{F\}$-modules, and we apply some results on right $R\{F\}$-modules to prove that when R is an F-finite regular local ring, every F_R-module has an F_R-injective resolution of length $\leq d+1$ where $d = \dim R$ (or equivalently, $\mathrm{Ext}^i_{F_R}(M, N) = 0$ for all F_R-modules M and N when $i > d+1$). In Section 4 we show some (non)finiteness results on $\mathrm{Ext}^1_{F_R}(M, N)$ when M and N are F_R-finite F_R-modules. Examples will be given throughout.

2. Preliminaries

Throughout this paper, $(R, \mathfrak{m}, K)$ will always denote a regular local ring of equal characteristic $p > 0$ and dimension d. We use $R^{(1)}$ to denote the target ring of the Frobenius map $F\colon R \to R$. When M is an R-module, we use $M^{(1)}$ to denote the corresponding module over $R^{(1)}$. We shall let F denote the Frobenius functor from R-modules to R-modules. In detail, $F(M)$ is given by base change to $R^{(1)}$ and then identifying $R^{(1)}$ with R. Note that by Kunz's result [Kun69], we know that $R^{(1)}$ is faithfully flat as an R-module. We say R is *F-finite* if $R^{(1)}$ is finitely generated as an R-module. So for an F-finite regular local ring, $R^{(1)}$ is finite and free as an R-module.

We use $R\{F\}$ to denote the Frobenius skew polynomial ring, which is the noncommutative ring generated over R by the symbols $1, F, F^2, \ldots$ by requiring that $Fr = r^pF$ for $r \in R$. Note that $R\{F\}$ is always free as a left R-module and flat as a right R-module. When R is F-finite, $R\{F\}$ is also free as a right R-module (because $R^{(1)}$ is finite free in this case). We say an R-module M is a *right $R\{F\}$-module* if it is a right module over the ring $R\{F\}$, or equivalently, there exists a morphism $\phi\colon M \to M$ such that for all $r \in R$ and $x \in M$, $\phi(r^px) = r\phi(x)$ (the right action of F can be identified with ϕ). This morphism can be also viewed as an R-linear map $\phi\colon M^{(1)} \to M$. We note that a right $R\{F\}$-module is the same as a *Cartier module* defined in [BB11] (where it is defined for general noetherian rings and schemes of equal characteristic $p > 0$).

We collect some definitions from [Lyu97]. These are the key concepts and objects that we shall study in this note. Notice that the original definitions (as well as most results in [Lyu97] and [Hoc07]) do not require that R be local. Nonetheless, we will restrict to the local case to avoid some technical arguments.

Definition 2.1 (*cf.* Definition 1.1 in [Lyu97])**.** An *F_R-module* is an R-module M equipped with an R-linear isomorphism $\theta\colon M \to F(M)$ which we call the structure morphism of M. A homomorphism of F_R-modules is an R-module homomorphism $f\colon M \to M'$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\theta"] & M' \arrow[d, "\theta'"] \\ F(M) \arrow[r, "F(f)"] & F(M') \end{tikzcd}$$

Definition 2.2 (*cf.* Definition 1.9 and Definition 2.1 in [Lyu97])**.** A generating morphism of an F_R-module M is an R-module homomorphism $\beta\colon M_0 \to F(M_0)$, where M_0 is some R-module, such that M is the limit of the inductive system in the top row of the commutative diagram

$$\begin{tikzcd} M_0 \arrow[r, "\beta"] \arrow[d, "\beta"] & F(M_0) \arrow[r, "F(\beta)"] \arrow[d, "F(\beta)"] & F^2(M_0) \arrow[r, "F^2(\beta)"] \arrow[d, "F^2(\beta)"] & \cdots \\ F(M_0) \arrow[r, "F(\beta)"] & F^2(M_0) \arrow[r, "F^2(\beta)"] & F^3(M_0) \arrow[r, "F^3(\beta)"] & \cdots \end{tikzcd}$$

and $\theta\colon M \to F(M)$, the structure isomorphism of M, is induced by the vertical arrows in this diagram. An F_R-module M is called *F_R-finite* if M has a generating morphism $\beta\colon M_0 \to F(M_0)$ with M_0 a finitely generated R-module.

We also recall some important Theorems in [Lyu97] that we shall need.

Theorem 2.3 (*cf.* Example 1.2(b$''$) in [Lyu97])**.** *If M is an F_R-module and $0 \to M \to I^\bullet$ is the minimal injective resolution of M in the category of R-modules, then each I^j acquires a structure of F_R-module such that the resolution becomes a complex of F_R-modules and F_R-module homomorphisms.*

Theorem 2.4 (*cf.* Proposition 2.9(b) in [Lyu97])**.** *If M is an F_R-module, so is M_f for every $f \in R$. Moreover, when M is an F_R-finite F_R-module, so is M_f.*

Theorem 2.5 (*cf.* Theorem 3.2 in [Lyu97])**.** *Every F_R-finite F_R-module has finite length in the category of F_R-modules (note that we assumed R is local).*

Theorem 2.6 (*cf.* Theorem 2.12(b) in [Lyu97])**.** *A simple F_R-module has a unique associated prime.*

3. Global dimension of Lyubeznik's F-modules

In this section we relate Lyubeznik's F_R-modules with right $R\{F\}$-modules (we need that R be regular here) and prove the main results on injective resolution of F_R-modules. We start by proving that the category of right $R\{F\}$-modules has finite global dimension $\leq d+1$ (note that in [EK04], a similar result on left $R\{F\}$-modules was proved and there the results were stated for schemes but only for finite Tor-dimension).

Lemma 3.1. *Let R be a regular local ring, and let M be a right $R\{F\}$-module, so that there is a map $\phi\colon M \to M$ such that $\phi(r^p x) = r\phi(x)$ for every $r \in R$ and $x \in M$ (note this can be also viewed as an R-linear map $\phi\colon M^{(1)} \to M$). Then we have an exact sequence of right $R\{F\}$-modules*

$$0 \to M^{(1)} \otimes_R R\{F\} \xrightarrow{\alpha} M \otimes_R R\{F\} \xrightarrow{\beta} M \to 0$$

where for every $x \in M^{(1)}$,

$$\alpha(x \otimes F^i) = \phi(x) \otimes F^i - x \otimes F^{i+1}$$

and for every $y \in M$,

$$\beta(y \otimes F^i) = \phi^i(y)$$

Proof. It is clear that every element in $M^{(1)} \otimes_R R\{F\}$ (resp. $M \otimes_R R\{F\}$) can be written uniquely as a (finite) sum $\sum x_i \otimes F^i$ where $x_i \in M^{(1)}$ (resp. $x_i \in M$) because $R\{F\}$ is free as a left R-module (this verifies that our maps α and β are well-defined). It is straightforward to check that α, β are morphisms of right $R\{F\}$-modules and that $\beta \circ \alpha = 0$ and β is surjective (because $\beta(y \otimes 1) = \phi^0(y) = y$). So it suffices to show α is injective and $\ker(\beta) \subseteq \operatorname{im}(\alpha)$.

Suppose $\alpha(\sum x_i \otimes F^i) = 0$. By definition of α we get $\sum(\phi(x_i) - x_{i-1}) \otimes F^i = 0$. Hence by uniqueness we get $\phi(x_i) = x_{i-1}$ for all i. Hence $x_i = 0$ for all i (because it is a finite sum). This proves α is injective.

Now suppose $\beta(\sum_{i=0}^n y_i \otimes F^i) = 0$. We want to find x_i $(0 \leq i \leq n)$ such that

$$\alpha(\sum_{i=0}^n x_i \otimes F^i) = \sum_{i=0}^n y_i \otimes F^i. \tag{3.1.1}$$

By definition of β we know that $\sum_{i=0}^{n} \phi^i(y_i) = 0$. Now one can check that

$$\begin{aligned}
&x_0 = -(y_1 + \phi(y_2) + \cdots + \phi^{n-1}(y_n)) \\
&x_2 = -(y_2 + \phi(y_3) + \cdots + \phi^{n-2}(y_n)) \\
&\cdots \\
&x_{n-1} = -y_n \\
&x_n = 0
\end{aligned}$$

is a solution of (3.1.1). This proves $\ker(\beta) \subseteq \operatorname{im}(\alpha)$. $\square$

Theorem 3.2. *Let R be an F-finite regular local ring. Then the category of right $R\{F\}$-modules has finite global dimension $\leq d+1$.*

Proof. It suffices to show that every right $R\{F\}$-module M has $R\{F\}$-projective dimension $\leq d+1$ in this category. We take a free resolution of M (in the category of right $R\{F\}$-modules) and truncate it at the d-th spot

$$0 \to P \to P_{d-1} \to \cdots \to P_0 \to M \to 0$$

where each P_i $(0 \leq i \leq d-1)$ is a free right $R\{F\}$-module. Since R is F-finite, $R\{F\}$ is free as a right R-module. The above resolution can be viewed as a free resolution of M over R. So we know that P is a right $R\{F\}$-module and is projective as an R-module.

By Lemma 3.1, we have an exact sequence of right $R\{F\}$-modules

$$0 \to P^{(1)} \otimes_R R\{F\} \to P \otimes_R R\{F\} \to P \to 0.$$

Since P is projective as an R-module and R is F-finite, $P^{(1)}$ is also projective as an R-module. So $P^{(1)} \otimes_R R\{F\}$ and $P \otimes_R R\{F\}$ are projective as right $R\{F\}$-modules. This shows that

$$0 \to P^{(1)} \otimes_R R\{F\} \to P \otimes_R R\{F\} \to P_{d-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resolution of M of length $d+1$ in the category of right $R\{F\}$-modules. $\square$

The next lemma basically tells that when R is F-finite, every F_R-module has a (non-canonical) structure of a right $R\{F\}$-module. We note that a large part of this lemma has already been observed in [LZZ11] where a more explicit description when R is a polynomial ring is given. We give a proof here for completeness and we also check functoriality in detail.

Lemma 3.3. *Let R be an F-finite regular local ring. We fix an isomorphism (as $R^{(1)}$-modules) ϕ: $\operatorname{Hom}_R(R^{(1)}, R) \to R^{(1)}$. Then giving an R-module M a right $R\{F\}$-module structure is equivalent to giving an R-linear morphism α_M: $M \to F(M)$. Moreover, giving an $R\{F\}$-linear map τ between two right $R\{F\}$-modules M, N is equivalent to giving a commutative diagram*

$$\begin{array}{ccc}
M & \xrightarrow{\tau} & N \\
\downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_N} \\
F(M) & \xrightarrow{1\otimes\tau} & F(N)
\end{array}$$

In particular, every F_R-module has a structure of right $R\{F\}$-module and maps between F_R-modules are maps of right $R\{F\}$-modules.

Proof. We have the (non-canonical) isomorphism ϕ because R is an F-finite regular local ring (hence $R^{(1)}$ is finite free over R). Notice that giving M a right $R\{F\}$-module structure is the same as giving an R-linear map

$$M^{(1)} \to M$$

which is the same as giving an $R^{(1)}$-linear map

$$M^{(1)} \to \mathrm{Hom}_R(R^{(1)}, M).$$

This in turn is the same as giving an $R^{(1)}$-linear map (because $R^{(1)}$ is finite free over R)

$$M^{(1)} \to \mathrm{Hom}_R(R^{(1)}, R) \otimes_R M.$$

Using the isomorphism ϕ: $\mathrm{Hom}_R(R^{(1)}, R) \to R^{(1)}$, we have that this is the same as giving an $R^{(1)}$-linear map

$$M^{(1)} \to R^{(1)} \otimes_R M.$$

But this is the same as a map (identifying $R^{(1)}$ with R)

$$M \to F(M).$$

Now we check functoriality. To give a map $M \xrightarrow{\tau} N$ of right $R\{F\}$-modules is the same as giving a commutative diagram

$$\begin{tikzcd} M^{(1)} \arrow[r, "\tau^{(1)}"] \arrow[d] & N^{(1)} \arrow[d] \\ \mathrm{Hom}_R(R^{(1)}, M) \arrow[r, "\mathrm{Hom}_R(R^{(1)},\tau)"] & \mathrm{Hom}_R(R^{(1)}, N) \end{tikzcd} \tag{3.3.1}$$

But we always have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_R(R^{(1)}, M) \arrow[r, "\mathrm{Hom}_R(R^{(1)},\tau)"] \arrow[d, "\cong"] & \mathrm{Hom}_R(R^{(1)}, N) \arrow[d, "\cong"] \\ \mathrm{Hom}_R(R^{(1)}, R) \otimes_R M \arrow[r, "1\otimes\tau"] \arrow[d, "\phi\otimes 1"] & \mathrm{Hom}_R(R^{(1)}, R) \otimes_R N \arrow[d, "\phi\otimes 1"] \\ R^{(1)} \otimes_R M \arrow[r, "1\otimes\tau"] & R^{(1)} \otimes_R N \end{tikzcd}$$

So giving a commutative diagram (3.3.1) is the same as giving

$$\begin{tikzcd} M \arrow[r, "\tau"] \arrow[d, "\alpha_M"] & N \arrow[d, "\alpha_N"] \\ F(M) \arrow[r, "1\otimes\tau"] & F(N) \end{tikzcd}$$

□

Remark 3.4. It is worth pointing out that every $R^{(1)}$-isomorphism ϕ': $\mathrm{Hom}_R(R^{(1)}, R) \to R^{(1)}$ is obtained from a fixed ϕ by multiplication by an invertible element of $R^{(1)}$, i.e. $\phi' = u \cdot \phi$ where $u \in R^{(1)}$ is a unit (see [LZZ11]). Therefore for any F_R-module M, the right $R\{F\}$-module structures induced by ϕ and ϕ' differ by multiplication by a unit in $R^{(1)}$.

Now we come to one of the main results in this note. Recall that by Theorem 1.1, every F_R-module has an injective resolution in this category (we call such resolution an F_R-injective resolution). We will show that each F_R-module has an F_R-injective resolution of length $\leq d+1$ (or equivalently $\mathrm{Ext}^i_{F_R}(M,N)=0$ for $i>d+1$). In some sense this proves that the category of F_R-modules has finite "global dimension".

Theorem 3.5. *Let R be an F-finite regular local ring, and let M, N be two F_R-modules. We fix an isomorphism ϕ: $\mathrm{Hom}_R(R^{(1)},R)\to R^{(1)}$ and give M, N right $R\{F\}$-module structures via ϕ as in Lemma 3.3. Then we have $\mathrm{Ext}^i_{F_R}(M,N)\cong \mathrm{Ext}^i_{R\{F\}}(M,N)$ for every i, where the left hand side denotes the* Ext *group computed in the category of F_R-modules and the right hand side denotes the* Ext *group computed in the category of right $R\{F\}$-modules.*

Proof. We use Yoneda's characterization of Ext^i (*cf.* Chapter 3.4 in [Wei94]). Note that this is the same as the derived functor Ext^i whenever the abelian category has enough injectives or enough projectives, hence holds for both the category of F_R-modules and the category of right $R\{F\}$-modules. An element in $\mathrm{Ext}^i_{F_R}(M,N)$ (resp. $\mathrm{Ext}^i_{R\{F\}}(M,N)$) is an equivalence class of exact sequences of the form

$$\xi : 0\to N\to X_1\to\cdots\to X_i\to M\to 0$$

where each X_i is an F_R-module (resp. right $R\{F\}$-module) and the maps are maps of F_R-modules (resp. maps of right $R\{F\}$-modules). The equivalence relation is generated by the relation $\xi_X\sim\xi_Y$ if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_i \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

From this characterization of Ext^i it is clear that we have a well-defined map

$$\iota : \mathrm{Ext}^i_{F_R}(M,N)\to \mathrm{Ext}^i_{R\{F\}}(M,N)$$

taking an equivalence class of an exact sequence of F_R-modules to the same exact sequence but viewed as an exact sequence in the category of right $R\{F\}$-modules (because we fix ϕ, every F_R-module has the structure of a right $R\{F\}$-module and this identification is functorial by Lemma 3.3).

Conversely, if we have an element in $\mathrm{Ext}^i_{R\{F\}}(M,N)$, say ξ, we have an exact sequence of right $R\{F\}$-modules, by Lemma 3.3 again, this induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & F(N) \arrow[r] \arrow[d, "\cong"] & F(X_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & F(X_i) \arrow[r] \arrow[d] & F(M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & F^2(N) \arrow[r] \arrow[d, "\cong"] & F^2(X_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & F^2(X_i) \arrow[r] \arrow[d] & F^2(M) \arrow[r] \arrow[d, "\cong"] & 0 \\
& \, & \, & & \, & \, &
\end{tikzcd}$$

Taking direct limits for columns and noticing that M, N are F_R-modules, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & N \arrow[r] & \varinjlim F^e(X_1) \arrow[r] & \cdots \arrow[r] & \varinjlim F^e(X_i) \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \tag{3.5.1}$$

It is clear that the bottom exact sequence represents an element in $\operatorname{Ext}^i_{F_R}(M,N)$, we call this element ξ'. Then we have a map

$$\eta : \operatorname{Ext}^i_{R\{F\}}(M,N) \xrightarrow{\xi \mapsto \xi'} \operatorname{Ext}^i_{F_R}(M,N).$$

This map is well-defined because it is easy to check that if $\xi_1 \sim \xi_2$, then we also have $\xi_1' \sim \xi_2'$. It is also straightforward to check that ι and η are inverses of each other. Obviously $\eta \circ \iota([\xi]) = [\xi]$ and $\iota \circ \eta([\xi]) = [\xi'] = [\xi]$, where the last equality is by (3.5.1) (which shows that $\xi \sim \xi'$, and hence they represent the same equivalence class in $\operatorname{Ext}^i_{R\{F\}}(M,N)$). $\square$

Remark 3.6. With the same assumptions and notations as in the above Theorem 3.5, if we use $\operatorname{Ext}^i_{R\{F\}}(M,-)|_{F_R}$ to denote $\operatorname{Ext}^i_{R\{F\}}(M,-)$ restricted to the category of F_R-modules, then we actually have $\operatorname{Ext}^i_{R\{F\}}(M,-)|_{F_R} \cong \operatorname{Ext}^i_{F_R}(M,-)$ *as functors* for every $i \geq 0$.

To see this, it suffices to show that $\operatorname{Ext}^i_{R\{F\}}(M,-)|_{F_R}$ is the i-th right derived functor of $\operatorname{Hom}_{F_R}(M,-)$ (for a detailed explanation of derived functors and universal δ-functor, we refer to [Har77], Chapter 3.1). First I claim that $(\operatorname{Ext}^i_{R\{F\}}(M,-)|_{F_R})_{i\geq 0}$ form a universal δ-functor. Obviously they form a δ-functor (because they form a δ-functor in the larger category of right $R\{F\}$-modules). This δ-functor is universal because for every F_R-injective module I, $\operatorname{Ext}^i_{R\{F\}}(M,I) \cong \operatorname{Ext}^i_{F_R}(M,I) = 0$ by Theorem 3.5. So by Corollary 1.4 in Chapter 3 of [Har77], $\operatorname{Ext}^i_{R\{F\}}(M,-)|_{F_R}$ is isomorphic to the i-th derived functor of $\operatorname{Ext}^0_{R\{F\}}(M,-)|_{F_R} = \operatorname{Hom}_{R\{F\}}(M,-)|_{F_R}$. But $\operatorname{Hom}_{R\{F\}}(M,-)|_{F_R} \cong \operatorname{Hom}_{F_R}(M,-)$ by Lemma 3.3. So $\operatorname{Ext}^i_{R\{F\}}(M,-)|_{F_R}$ is the i-th derived functor of $\operatorname{Hom}_{F_R}(M,-)$, which is (by definition) $\operatorname{Ext}^i_{F_R}(M,-)$ for every i.

Theorem 3.7. *Let R be an F-finite regular local ring. Then each F_R-module has an F_R-injective resolution of length $\leq d+1$.*

Proof. By Theorems 3.2 and 3.5, it is clear that for each pair of F_R-modules N_1 and N_2, $\operatorname{Ext}^i_{F_R}(N_1,N_2) = \operatorname{Ext}^i_{R\{F\}}(N_1,N_2) = 0$ when $i > d+1$. Now the Theorem follows from standard homological algebra argument. For any F_R-module M, take an F_R-injective resolution and truncate it at the $(d+1)$-th spot, we get an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_d \to N \to 0$$

where each E_i $(0 \leq i \leq d)$ is injective as an F_R-module. Now for any F_R-module L, we have $\operatorname{Ext}^1_{F_R}(L,N) = \operatorname{Ext}^{d+2}_{F_R}(L,M) = \operatorname{Ext}^{d+2}_{R\{F\}}(L,M) = 0$. This proves that N is injective as an F_R-module and hence M has an F_R-injective resolution of length $d+1$. $\square$

Now we study some examples. The simplest example of an F_R-module is R equipped with structure isomorphism the identity map (sending 1 in R to 1 in $F(R) \cong R$). Another important example is the injective hull of the residue field $E = E(R/\mathfrak{m})$. We can give it

a generating morphism β: $R/\mathfrak{m} \to F(R/\mathfrak{m})$ by sending $\overline{1}$ to $\overline{x_1^{p-1} \cdots x_n^{p-1}}$ (where $x_1, \dots, x_n$ are minimal generators of $\mathfrak{m}$). We will call these structure isomorphisms of R and E the *standard* F_R-module structure on R and E.

Example 3.8. Let's look at the minimal injective resolution of R (in the category of R-modules)

$$0 \to R \to E(R) \to \oplus_{\mathrm{ht}(P)=1} E(R/P) \to \cdots \to \oplus_{\mathrm{ht}(P)=d-1} E(R/P) \to E \to 0.$$

If we give R the standard F_R-module structure, the above sequence is an exact sequence of F_R-modules also by Theorem 2.3. Hence by Yoneda's characterization of Ext groups, it represents an element in $\mathrm{Ext}^d_{F_R}(E, R)$. It is clear that this element is not 0 because it is not 0 even in $\mathrm{Ext}^d_R(E, R)$ ($\mathrm{Ext}^d_R(E, R) \cong \mathrm{Hom}_R(E, E)$ and it is easy to check that the above sequence as an element in $\mathrm{Ext}^d_R(E, R)$ corresponds to the identity map in $\mathrm{Hom}_R(E, E)$).

Remark 3.9. Example 3.8 shows that the "global dimension" of the category of F_R-modules is at least d. Hence, combined with our Theorem 3.7, when R is F-finite this global dimension is either d or $d + 1$. We conjecture that it is $d + 1$. When $R = K$ is an F-finite field, the next example will show that this global dimension is indeed 1 (not $0 = \dim K$).

Example 3.10. Let $M = \oplus_{i \in \mathbb{Z}} R x_i$ denote the infinite direct sum of copies of R. We can give M an F_R-module structure by setting

$$\theta_M : x_i \to x_{i+1}.$$

One can check that we have an F_R-linear map γ: $M \to R$ (the target R is equipped with the standard F_R-module structure) such that

$$\gamma(\sum r_i x_i) = \sum r_i.$$

Call the kernel of this map N, then we have an exact sequence of F_R-modules

$$0 \to N \to M \xrightarrow{\gamma} R \to 0.$$

I claim that the above sequence does not split (in the category of F_R-modules). Suppose g: $R \to M$ is a splitting, say $g(1) = y \neq 0$. Then a direct computation gives that $\theta_M(y) = y^p$, which is impossible by the definition of θ_M. Hence, by Yoneda's characterization of Ext groups, we know that $\mathrm{Ext}^1_{F_R}(R, N) \neq 0$. In particular the above holds when we take R to be any F-finite field K, so the global dimension of the category of F_K-modules is 1.

4. non-Finiteness of $\mathrm{Ext}^1_{F_R}$

Recall that by theorem 1.2, $\mathrm{Hom}_{F_R}(M, N)$ is a finite set when M and N are F_R-finite F_R-modules. We prove that $\mathrm{Ext}^1_{F_R}(M, N)$ is finite when M is supported only at $\mathfrak{m}$ and R has separably closed residue field. And we show that in general $\mathrm{Ext}^1_{F_R}(M, N)$ is not necessarily a finite set. We need some lemmas.

Lemma 4.1 (*cf.* Proposition 3.1 in [Lyu97]). *Let S be a regular local ring of equal characteristic $p > 0$ and let $R \to S$ be a surjective homomorphism with kernel $I \subseteq R$. There exists an equivalence of categories between the F_R-modules supported on* $\mathrm{Spec}\, S = V(I) \subseteq \mathrm{Spec}\, R$ *and the F_S-modules. Under this equivalence the F_R-finite F_R-modules supported on* $\mathrm{Spec}\, S = V(I) \subseteq \mathrm{Spec}\, R$ *correspond to the F_S-finite F_S-modules.*

Lemma 4.2 (*cf.* Theorem 4.2(c)(e) in [Hoc07]). *Let K be a separably closed field. Then every F_K-finite F_K-module is isomorphic with a finite direct sum of copies of K with the standard F_K-module structure. Moreover,* $\mathrm{Ext}^1_{F_K}(K, K) = 0$.

Lemma 4.3. *Let $(R, \mathfrak{m}, K)$ be a regular local ring with K separably closed. Then every F_R-finite F_R-module supported only at $\mathfrak{m}$ is isomorphic (as F_R-module) with a finite direct sum of copies of $E = E(R/\mathfrak{m})$ (where E is equipped with the standard F_R-module structure). Moreover,* $\mathrm{Ext}^1_{F_R}(E, E) = 0$.

Proof. This is clear from Lemma 4.1 (applied to $S = K$ and $I = \mathfrak{m}$) and Lemma 4.2 (it is straightforward to check that the standard F_R-module structure on E correspond to the standard F_K-module structure on K). $\square$

Theorem 4.4. *Let $(R, \mathfrak{m}, K)$ be a regular local ring such that K is separably closed and let M, N be F_R-finite F_R-modules. Then $\mathrm{Ext}^1_{F_R}(M, N)$ is finite if M is supported only at $\mathfrak{m}$.*

Proof. Since K is separably closed, by Lemma 4.3 we know that M is a finite direct sum of copies of E in the category of F_R-modules. So it suffices to show that $\mathrm{Ext}^1_{F_R}(E, N)$ is finite. For

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

exact, we know that

$$\mathrm{Ext}^1_{F_R}(E, N_1) \to \mathrm{Ext}^1_{F_R}(E, N_2) \to \mathrm{Ext}^1_{F_R}(E, N_3)$$

is exact. So we immediately reduce to the case that N is simple (since we assume local, every F_R-finite F_R-module has finite length by Theorem 2.5).

We want to show that $\mathrm{Ext}^1_{F_R}(E, N)$ is finite when N is simple. There are two cases: $\mathrm{Ass}_R(N) = \mathfrak{m}$ or $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$ (by Theorem 2.6). If $\mathrm{Ass}_R(N) = \mathfrak{m}$, then $N \cong E$ as F_R-module by Lemma 4.3. Then $\mathrm{Ext}^1_{F_R}(E, N) = \mathrm{Ext}^1_{F_R}(E, E) = 0$ by Lemma 4.3.

If $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$, by Yoneda's characterization of Ext groups, it suffices to show that we only have a finite number of isomorphism classes of short exact sequences

$$0 \to N \to L \to E \to 0$$

of F_R-modules. We first show the number of choices of isomorphism classes for L is finite. Say $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$, then we have $P \in \mathrm{Ass}_R(L) \subseteq \{P, \mathfrak{m}\}$. If $\mathrm{Ass}_R(L) = \{P, \mathfrak{m}\}$, then $H^0_{\mathfrak{m}}(L) \neq 0$ and it does not intersect N. So $H^0_{\mathfrak{m}}(L) \oplus N$ is an F_R-submodule of L. Hence we must have $L \cong H^0_{\mathfrak{m}}(L) \oplus N \cong E \oplus N$ (since L has length 2 as F_R-module). If $\mathrm{Ass}_R(L) = \{P\}$, we can pick $x \in \mathfrak{m} - P$. Localizing at x gives a short exact sequence

$$0 \to N_x \to L_x \to E_x \to 0.$$

But $E_x = 0$, so we get $N_x \cong L_x$ as F_R-module. Since x is not in P, we have $L \hookrightarrow L_x$ as F_R-module. That is, L is isomorphic to an F_R-submodule of L_x, hence is isomorphic to an F_R-submodule of N_x. But N_x is F_R-finite by Theorem 2.4, so it only has finitely many F_R-submodules by Theorem 1.2. This proves that the number of choices of isomorphism classes for L is finite.

Because the number of choices of isomorphism classes for L is finite and (for each F_R-finite F_R-module L) $\mathrm{Hom}_{F_R}(N, L)$ is always finite by Theorem 1.2. Hence the number of isomorphism classes of short exact sequences $0 \to N \to L \to E \to 0$ is finite. $\square$

If M is an F_R-module with structure morphism θ_M, for every $x \in M$ we use x^p to denote $\theta_M^{-1}(1 \otimes x)$. Notice that when $M = R$, this is exactly the usual meaning of x^p. We let G_M denote the set $\{x^p - x | x \in M\}$. It is clear that G_M is an abelian subgroup of M.

Theorem 4.5. *Giving R the standard F_R-module structure, then for every F_R-module M we have $\mathrm{Ext}^1_{F_R}(R, M) \cong M/G_M$ as an abelian group.*

Proof. By Yoneda's characterization of Ext groups, an element in $\mathrm{Ext}^1_{F_R}(R, M)$ can be represented by an exact sequence of F_R-modules

$$0 \to M \to L \to R \to 0.$$

It is clear that $L \cong M \oplus R$ as R-module. Moreover, one can check that the structure isomorphism θ_L composed with $\theta_M^{-1} \oplus \theta_R^{-1}$ defines an isomorphism

$$M \oplus R \xrightarrow{\theta_L} F(M) \oplus F(R) \xrightarrow{\theta_M^{-1} \oplus \theta_R^{-1}} M \oplus R$$

which sends (y, r) to $(y + rz, r)$ for every $(y, r) \in M \oplus R$ and for some $z \in M$. Hence, giving a structure isomorphism of L is equivalent to giving some $z \in M$. Therefore, θ_L is determined by an element $z \in M$. Two exact sequences with structure isomorphism θ_L, θ'_L are in the same isomorphism class if and only if there exists a map $g \colon L \to L$, sending (y, r) to $(y + rx, r)$ for some $x \in M$ such that

$$(1 \otimes g) \circ \theta_L = \theta'_L \circ g.$$

Now we apply $\theta_M^{-1} \oplus \theta_R^{-1}$ on both side. If θ_L, θ'_L are determined by z_1 and z_2 respectively, a direct computation gives that

$$(\theta_M^{-1} \oplus \theta_R^{-1}) \circ (1 \otimes g) \circ \theta_L(y, r) = (y + rz_1 + rx^p, r)$$

while

$$(\theta_M^{-1} \oplus \theta_R^{-1}) \circ \theta'_L \circ g(y, r) = (y + rz_2 + rx, r).$$

So θ_L and θ'_L are in the same isomorphism class if and only if there exists $x \in M$ such that

$$z_2 - z_1 = x^p - x.$$

So $\mathrm{Ext}^1_{F_R}(R, M) \cong M/G_M$ as an abelian group. $\square$

Corollary 4.6. *Let $(R, \mathfrak{m}, K)$ be a complete regular local ring with K separably closed. Then* $\mathrm{Ext}^1_{F_R}(R, R) = 0$

Proof. By Theorem 4.5, it suffices to show $R = G_R$. That is, for every $r \in R$ we can find $x \in R$ such that $x^p - x = r$. Since we have an exact sequence

$$0 \to \mathfrak{m} \to R \to K \to 0$$

so one sees easily that to solve the equation in R, it is sufficient to solve the equation in $\mathfrak{m}$ and K. One can solve it in K because K is separably closed. For every $y \in \mathfrak{m}$, we let

$$x = -y - y^p - y^{p^2} - \cdots$$

which is well-defined because $y \in \mathfrak{m}$ and R is complete. One can check easily that $x^p - x = y$ holds. This finishes the proof. $\square$

Now we give some examples to show that, in general, $\mathrm{Ext}^1_{F_R}(R, M) \cong M/G_M$ is not necessarily finite, even in simple cases.

Example 4.7. Let $R = k(t)$ or $k[t]_{(t)}$ with k an algebraically closed field. Hence in the first case, R is a non-separably closed F-finite field and in the second case, R is a non-complete F-finite DVR with algebraically closed residue field. We will prove that $\mathrm{Ext}^1_{F_R}(R,R)$ is infinite in both cases (this will show that the conditions in Corollary 4.6 are necessary). By Theorem 4.5, it suffices to show that for $a, b \in k$ ($a, b \neq 0$ in the second case), $\dfrac{1}{t-a}$ and $\dfrac{1}{t-b}$ are different in R/G_R whenever $a \neq b$. Otherwise there exists $\dfrac{h(t)}{g(t)} \in R$ with $h(t), g(t) \in k[t]$ ($g(t)$ is not divisible by t in the second case) and $\gcd(h(t), g(t)) = 1$ such that

$$\frac{1}{t-a} - \frac{1}{t-b} = \frac{h(t)^p}{g(t)^p} - \frac{h(t)}{g(t)}$$

which gives

$$\frac{a-b}{t^2-(a+b)t+ab} = \frac{h(t)^p - h(t)\cdot g(t)^{p-1}}{g(t)^p}. \tag{4.7.1}$$

Since $\gcd(h(t), g(t)) = 1$, $\gcd(h(t)^p - h(t)\cdot g(t)^{p-1}, g(t)^p) = 1$. So from (4.7.1) we know that $g(t)^p | (t^2 - (a+b)t + ab)$. This is clearly impossible.

Example 4.8. Let $(R, \mathfrak{m}, K)$ be a regular local ring with K infinite, and let $E = E(R/\mathfrak{m})$ be the injective hull of the residue field. We will show that $\mathrm{Ext}^1_{F_R}(R, E)$ is always infinite when $\dim R \geq 1$. Recall that $E = \varinjlim_n \dfrac{R}{(x_1^n, \ldots, x_d^n)}$ so every element z in E can be expressed as $(r; x_1^n, \ldots, x_d^n)$ for some $n \geq 1$ (which means z is the image of r in the n-th piece in this direct limit system).

By Theorem 4.5 it suffices to show that E/G_E is infinite. I claim that two different socle elements u_1, u_2 are different in E/G_E. Otherwise we have

$$u_1 - u_2 = z^p - z \tag{4.8.1}$$

in E. Since $u_1 - u_2$ is a nonzero element in the socle of E, we may write $u_1 - u_2 = (\lambda; x_1, \ldots, x_d)$ in E (for some $\lambda \neq 0$ in K). Say $z = (r; x_1^n, \ldots, x_d^n)$ with n minimum. Then (4.8.1) will give

$$(r; x_1^n, \ldots, x_d^n) = (\lambda; x_1, \ldots, x_d) + (r^p; x_1^{np}, \ldots, x_d^{np}).$$

This will give us

$$r^p + \lambda(x_1 \cdots x_d)^{np-1} - r(x_1 \cdots x_d)^{np-n} \in (x_1^{np}, \ldots, x_d^{np}). \tag{4.8.2}$$

If $n = 1$, then $z \in \mathrm{Soc}(E)$, hence r is a unit in R (since $z \neq 0$). But (4.8.2) shows that $r^p \in (x_1, \ldots, x_d)$ which is a contradiction.

If $n \geq 2$, we have $np - 1 \geq np - n \geq p$. We know from (4.8.2) that for every $1 \leq i \leq d$, we have $r^p \in (x_1^{np}, \ldots, x_{i-1}^{np}, x_i^p, x_{i+1}^{np}, \ldots, x_d^{np})$. Hence $r \in (x_1^n, \ldots, x_{i-1}^n, x_i, x_{i+1}^n, \ldots, x_d^n)$ for every $1 \leq i \leq d$. Taking their intersection, we get that $r \in (x_1 \cdots x_d, x_1^n, \ldots, x_d^n)$. That is, mod $(x_1^n, \ldots, x_d^n)$, we have $r = (x_1 \cdots x_d) r_0$. But then we have $z = (r_0; x_1^{n-1}, \ldots, x_d^{n-1})$ contradicting our choice of n.

So we know that any two different socle elements u_1, u_2 are different in E/G_E. Since the socle of E is a one-dimensional K-vector space and we assume that K is infinite, this shows that E/G_E is infinite.

Remark 4.9. Example 4.8 also shows that $E = E(R/\mathfrak{m})$, though injective as an R-module, is *not* injective as an F_R-module (with its standard F_R-structure) when $\dim R \geq 1$.

Acknowledgement

I would like to thank Mel Hochster for many helpful and valuable discussions on the problem. I would also like to thank Gennady Lyubeznik for reading a preliminary version of the paper and for his comments.

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